Basic Notions:

A topological space $(X,\mathcal{T})$ is a set X & a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

- Closed under finite intersections
- Closed under arbitrary unions
- $X, \emptyset \in \mathcal{T}$

Mapping Cone:

$Cyl(X) = X \times I$

$$Cone(X) = Cyl(X) / (x,1) \sim (x',1)$$

For $f: X \to Y$

$$cone(f) = Y \amalg cone(X) / fx \sim (x,0)$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"] & Y \arrow[d] \\ cone(X) \arrow[r] & cone(f) \end{tikzcd}$$

$Y \subseteq X$ has subspace top given by
$\mathcal{T}_Y = \{ \mathcal{U} \cap Y : \mathcal{U} \in \mathcal{T}_X \}$.

Quotient Space: $Y = X/\sim$ $\quad x \sim x' \Leftrightarrow f(x) = f(x')$

$$\begin{tikzcd} X \arrow[r, "f", "\text{surjective}"'] \arrow[dr, "g"'] & Y \arrow[d, "h \; \exists! \text{ cts.}"] \\ & Z \end{tikzcd}$$

$\mathcal{T}_Y = \{ \mathcal{U} : f^{-1}(\mathcal{U}) \in \mathcal{T}_X \}$

$\leftarrow$ Universal property

g cts $x \sim x' \Rightarrow g(x) = g(y)$

Homotopy:

$X \simeq Y \Leftrightarrow \exists f: X \to Y$ & $\exists g: Y \to X$
$g \circ f \simeq id_X$ & $f \circ g \simeq id_Y$

X & Y top spaces

$f, g: X \to Y$ cts functions

$f \simeq g \Leftrightarrow \exists H: X \times [0,1] \to Y$
$H(x,0) = f(x)$
$H(x,1) = g(x)$

$A \subseteq X$ $\quad f \simeq_A g$
$\Leftrightarrow \forall a \in A \;\; \forall t \in [0,1] \;\; H(a,t) = f(a) = g(a)$

$\boxed{\mathbb{D}^n \simeq *}$

$f: * \to \mathbb{D}^n$, $\; * \mapsto 0$ $\qquad g \circ f \simeq id_*$ clear

$g: \mathbb{D}^n \to *$, $\; x \mapsto *$ $\qquad f \circ g: \mathbb{D}^n \to \mathbb{D}^n$, $\; v \mapsto 0$

$H: \mathbb{D}^n \times I \to \mathbb{D}^n$, $\; (v,t) \mapsto t \cdot v$

is a homotopy $f \circ g \simeq id_{\mathbb{D}^n}$.

Fundamental Group:

Let (X, x_0) a pointed Top space.

$$\pi_1(X) = \left(\underbrace{cts_{x_0}([0,1], X) / \simeq_{\{0,1\}}}, * \right)$$

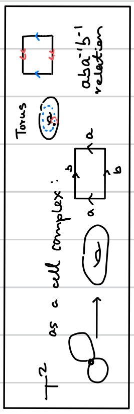
continuous basepoint preserving maps from $I \to X$ modulo homotopy on the boundary.

$*$ is the group operation concatenation

$$\gamma_1 * \gamma_2 = \begin{cases} \gamma_1(2t) & , t \in [0, \tfrac{1}{2}] \\ \gamma_2(2t-1) & , t \in [\tfrac{1}{2}, 1] \end{cases}$$

Paths have inverse $\gamma^{-1}(t) = \gamma(1-t)$

A space is simply connected $\Leftrightarrow \pi_1(X, x_0) = 1$. $\}$ eg. contractable spaces



π_1 as a functor:

$$\mathcal{T}op_* \xrightarrow{\pi_1} \mathcal{G}$$

Pointed spaces with cts base point preserving maps $\qquad$ Groups & group homomorphisms

$(X, x_0) \longmapsto \pi_1(X, x_0)$

$f \longmapsto f_*$

Where $f: X \to Y$ $\; f(x_0) = y_0$ we get that

$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$
$[\gamma] \longmapsto [f \circ \gamma]$

$(f \circ g)_* = f_* \circ g_*$ & $id_* = id$

$\mathbb{RP}^n \cong S^n / -x \sim x$
$\cong cone(p_{n-1}: S^{n-1} \to \mathbb{RP}^{n-1})$
$\cong \mathbb{R}^n \amalg \mathbb{RP}^{n-1}$

$\mathbb{RP}^n$ is the class of lines through the origin of $\mathbb{R}^{n+1}$, so we denote an element as $[x_1 : \cdots : x_{n+1}]$ (scale invariant)

Special case: $\mathbb{RP}^3 \cong SO(3)$

$SO(3) \cong M_{3\times3}(\mathbb{R})|_{det=1}$
$\cong$ Rotations in $\mathbb{R}^3$

$S^3 \subseteq \mathbb{H}$ quarternions:

$S^3 \xrightarrow[\text{quotient } \{\pm 1\}]{} \mathbb{RP}^3 \cong SO(3)$, $\eta = \cos\theta + (\sin\theta)\vec{u}$ $\quad$ purely imaginary

$\eta(-)\eta^{-1}$ $\quad \eta^{-1} \to \cos\theta - \sin\theta\,\vec{u}$

Seifert Van Kampen:

Free product of two groups $G * H$ is the group of reduced words of elements in G & H.
Group action is concatenation (then reduction).

This is a coproduct:

$$\begin{tikzcd} & G * H \arrow[d, dashed, "\exists!"] & \\ G \arrow[ur, "\iota"] \arrow[r, "f"'] & T & H \arrow[ul, "\iota"'] \arrow[l, "g"] \end{tikzcd}$$

For topological spaces the coproduct is wedge $X \vee Y = X \amalg Y / x_0 \sim y_0$
Identify basepoint

Amalgamated product is free group quotient an equivilence relation such that
$\alpha(k) \sim \beta(k)$
(as elements of $G * H$)

$$\begin{tikzcd} K \arrow[r, "\beta"] \arrow[d, "\alpha"] & G \arrow[d, "\iota"] \arrow[ddr, bend left, "f"] & \\ H \arrow[r, "\iota"] \arrow[drr, bend right, "g"'] & G *_K H \arrow[dr, dashed] & \\ & & T \end{tikzcd}$$

Note $K = 1 \Rightarrow G *_K H = G * H$

$$G *_K H = \left\langle \text{generators of } G \,\&\, H \;\middle|\; \begin{array}{l} \cdot\text{relations of } G \\ \cdot\text{relations of } H \\ \cdot\text{Amalgamation relations} \end{array} \right\rangle$$

The amalgamation relations are given by α & β above, there is one relation for each $k \in K$ given by $\iota\alpha(k) = \iota\beta(k)$
As elements of $G *_K H$.

SVK Theorem: (X, x_0) top space

- $\mathcal{U}, V \subseteq X$ open
- $X = \mathcal{U} \cup V$
- $\mathcal{U} \cap V$ path connected
- $x_0 \in \mathcal{U} \cap V$

$$\Rightarrow \pi_1(X, x_0) \cong \pi_1(\mathcal{U}, x_0) *_{\pi_1(\mathcal{U} \cap V, x_0)} \pi_1(V, x_0)$$

$$X \wedge Y = X \times Y / X \vee Y$$

$$\Sigma X = S^1 \wedge X$$

$$H_*(\Sigma X; pt) \cong H_{*-1}(X; pt)$$

Covering Spaces:

Assume all spaces are

- Hausdorff
- Arcwise connected → $\forall x,y \in X\ \exists \lambda : [0,1] \to X$, $\lambda(0) = x$, $\lambda(1) = y$
- Locally arcwise connected → $\forall x \in X\ \forall$ Neighbourhoods of x $\exists$ arcwise connected neighbourhood inside.

$p : X \to Y$ is a covering

$\Leftrightarrow \forall y \in Y\ \exists \mathcal{U} \in \mathcal{T}_y$ with

$$p^{-1}(\mathcal{U}) = \coprod_{\alpha \in A} V_\alpha$$

The V_α are path components of $p^{-1}(\mathcal{U})$ — Maximal arcwise connected subset.

& $p|_{V_\alpha}$ is a homeomorphism

Lifting Theorems

Path Lifting:

$$\begin{tikzcd} \{0\} \arrow[r, hook, "x_0"] \arrow[d] & X \arrow[d, "p"] \\ I \arrow[r, "f"'] \arrow[ur, dashed, "\exists! \tilde{f}"] & Y \end{tikzcd}$$

- $p : X \to Y$ cover
- $f : I \to Y$ path
- $x_0 \in X$ $\quad p(x_0) = f(0)$

$\Rightarrow \exists! \tilde{f} : I \to X$

$\tilde{f}(0) = x_0$, $f = p \circ \tilde{f}$

Covering Homotopy:

$$\begin{tikzcd} W \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & X \arrow[d, "p"] \\ W \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists! \tilde{F}"] & Y \end{tikzcd}$$

- W locally connected
- $p : X \to Y$ cover
- $\tilde{f} : W \times \{0\} \to X$ a lifting of $F|_{W \times \{0\}}$

$\Rightarrow \exists! \tilde{F} : W \times I \to X$ $\quad F = p \circ \tilde{F}$

If F is a homotopy rel $W' \subset W$ so is $\tilde{F}$

Alternitively:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

If $\exists w \in W.\ \tilde{f}_1(w) = \tilde{f}_2(w)$

then $\tilde{f}_1 = \tilde{f}_2$

(uniqueness of cts map lift)

Corrollories:

- f_0 & f_1 paths in Y
- $f_0 \simeq_{\{0,1\}} f_1$ $\quad$ • $\tilde{f}_0(0) = \tilde{f}_1(0)$

$\Rightarrow \tilde{f}_0 \simeq_{\{0,1\}} \tilde{f}_1$

Fixing a point makes the lifts unique.

Echidna:

Any cts vector field on S^2 has a zero.

Invariance of Dimension:

M, N manifolds of dim m & n respectively

$\mathcal{U} \subset M$, $V \subset N$ open (nonempty)

If $\exists$ a homeomorphism $\mathcal{U} \to V$ then $m = n$.

- $f : S^1 \to Y$ • $f \simeq_{\{*\}} \mathrm{id}$

$\Rightarrow \tilde{f}$ also a loop & $\tilde{f} \simeq_* \mathrm{id}$

$p_* : \pi_1(X, x_0) \to \pi_1(Y, p(x_0))$

is • injective • $\ker(p_*) = \{0\}$

• $\mathrm{Im}(p_*) = \{$loops in Y that lift to loops in $X\}$

Lifting to a loop is a property of the class $[f] \in \pi_1(Y, y_0)$

If f lifts to a loop in X then everything in $[f]$ does

Y has nontrivial covering space

$\Rightarrow \pi_1(Y, y_0) \neq 1$

↳ $\mathbb{RP}^n$ • T^2 • S^1 • Klein

$$\begin{tikzcd} W_1 \arrow[rr, dashed, "g"] \arrow[dr, "p_1"'] & & W_2 \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

If $p_1(\omega_1) = p_2(\omega_2)$

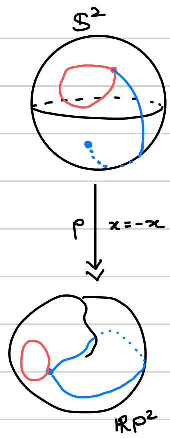
$\exists!\ g : W_1 \to W_2$

$g(\omega_1) = \omega_2$ & $p_2 \circ g = p_1$

W_1 simply cnct $\Rightarrow g$ is covering

W_1 & W_2 simply cnct $\Rightarrow g$ is homeomorphism

$\pi_1(\mathbb{RP}^2)$ by Covering:



A loop in $\mathbb{RP}^2$ can come from either a loop in S^2 or a path from x to $-x$.

So up to homotopy there are only two different paths.

So $\pi_1(\mathbb{RP}^2) = \{\pm 1\} \cong \mathbb{Z}_2$

To make this formal need the theory of universal covers.

$\pi_1(\mathbb{RP}^2)$ by SVK:

$\mathbb{RP}^2 = \underset{\text{Mobius band}}{M} \cup_{S^1} \underset{\text{glued along bandry}}{D^2}$ Note we have already use homotopy here to reduce the open overlapping sets to simpler equivilent sets.

SVK $\Rightarrow \pi_1(\mathbb{RP}^2) = \pi_1(M) \bigstar_{\pi_1(S^1)} \pi_1(D^2)$

$= \mathbb{Z} \star_{\mathbb{Z}} 1$ ($\times 2$, $x \mapsto 1$) $\Rightarrow$ relation $2x = 1$.

$\cong \mathbb{Z}/2\mathbb{Z}$.

For a cover $X \xrightarrow{p} Y$; The fiber over a point $y \in Y$ is $p^{-1}(y)$.

$\pi_1(Y,y_0)$ acts on the fiber. $\varphi \in p^{-1}(y)$

$[\gamma] \in \pi_1(Y,y_0) \Rightarrow \varphi \cdot [\gamma] = \tilde{\gamma}_\varphi(1)$

Where $\tilde{\gamma}_\varphi$ is the (unique) lift of γ starting at φ.

This action is transitive (one orbit $= F$)

Its stabilizer (of $x_0 \in X$) $p_*(\pi_1(X,x_0)) \subseteq \pi_1(Y,y_0)$

$\Rightarrow |F| = [\pi_1(Y,y_0) : p_*\pi_1(X,x_0)]$

Let $G \curvearrowright X$

Orbit of $x \in X$ is $G\cdot x = \{g\cdot x : g \in G\}$

Stabilizer of $x \in X$ $G_x = \{g : g\cdot x = x\}$

Recall the orbits partition X.

Deck Transformations:

A map $D: X \to X$ such that $p \circ D = p$.

$$\begin{tikzcd} X \arrow[rr, "D"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

- $\mathrm{Deck}(p) = \mathrm{Aut}(X/Y)$
- $D^{-1} \in \mathrm{Deck}(p)$
- D is always invertible.
- If $\exists x \in X$ $D(x) = x \Rightarrow D = id$.

$D \in \mathrm{Deck}(p)$, $[\alpha] \in \pi_1(Y,y_0)$, $x \in p^{-1}(y_0)$

$\Rightarrow D(x\cdot[\alpha]) = D(x)\cdot[\alpha]$

Deck tr' commute with the action of $\pi_1(Y,y_0) \curvearrowright F$ on the fiber.

The action of a discrete group G on a top' space X is properly discontinuous $\iff \forall x \in X\ \exists \mathcal{U} \subseteq X$ open $\boxed{\text{st. } \forall g \in G \quad g\mathcal{U} \cap \mathcal{U} \neq \emptyset \Rightarrow g = e}$

$G \curvearrowright X$ prop' Disc

Then $p: X \longrightarrow G\backslash X$ = orbits with quotient top

the quotient map p is a cover

- If in addition X is simply connected $\pi_1(G\backslash X) \cong G$

Normalizers: $H < G$, $N(H) = \{\eta \in G : \eta H \eta^{-1} = H\}$

$\exists D \in \mathrm{Deck}(p)$ $D(x_0) = x$

$\iff \exists [\alpha] \in N(p_*\pi_1(X,x_0))$, $x = x_0\cdot[\alpha]$

$\iff p_*\pi_1(X,x_0) = p_*\pi_1(X,x)$

Lusternik-Schnirelmann:
S^n covered by $A_1, \dots, A_{n+1}$ close sets
$\Rightarrow \exists i, \exists x$ $x, -x \in A_i$
(one of the closed sets contains a pair of antipodal points).

$p_*\pi_1(X,x_0) \trianglelefteq \pi_1(Y,y_0)$ (normal) $\iff p$ is regular $\iff \mathrm{Deck}(p) \curvearrowright F$ simply trans'

$p_*(\pi_1(X,x))$ $x \in X$ ranges over all conjugates of $p_*(\pi_1 X, x) \subseteq \pi_1(Y,y_0)$

$\forall x,y \in F$ $\exists!\, d \in \mathrm{Deck}(p)$ $p(x) = y$

Thm: $p_*\pi_1(X, x_0\cdot[\alpha]) = [\alpha]^{-1} p_*\pi_1(X,x_0)[\alpha]$

We have the following short exact sequence:

$1 \longrightarrow p_*\pi_1(X,x_0) \longrightarrow N(p_*(\pi_1(X,x_0))) \longrightarrow \mathrm{Deck}(p) \longrightarrow 1$

$\alpha \longmapsto d$ ($\exists! d$ such that $D(x_0) = x_0\cdot[\alpha]$)

p regular $\Rightarrow \mathrm{Deck}(p) \cong \pi_1(Y,y_0) / p_*\pi_1(X,x_0)$

$p: X \to Y$ a cover & $\pi_1(X,x_0) = 1$

$\Rightarrow \mathrm{Deck}(p) \cong \pi_1(Y,y_0)$

In this case p is a "universal cover".

Y has simply connected covering $\tilde{Y}$

$\Rightarrow$ Equivalence classes of covering spaces of Y (base point preserving cover) Are bijectively related to subgroups of $\pi_1(Y,y_0)$

classes without basepoint are given by conjugacy classes of subgroups $\pi_1(Y,y_0)$

$G = Fr(\alpha_1, \dots, \alpha_n)$ $H \leq G$ index p $\Rightarrow H = Fr($on $pn - p + 1$ gen$)$

The number of cosets ↳ $|\{gH : g \in G\}| = |G:H|$

Recall $|G| = |G:H||H|$.

X semilocally 1 connected / relatively simply connected:

$\forall x \in X\ \exists \mathcal{U} \subseteq X$ open $\pi_1(\mathcal{U}, x) = \{1\}$

X has universal cover $\iff$ X relatively simply connected

Lens Space: Let $S^{2n-1} \subset \mathbb{C}^n$ $(n \geq 2)$

Then for p prime, $\zeta = e^{2\pi i/p}$ the primitive p^{th} root of 1 & $q_1, \dots, q_n \in \mathbb{Z}$ rel' prime to p.

Then $G = \langle \zeta \rangle$ = cyclic group of p elements $\subseteq \mathbb{C}$

and we can embed $\zeta \longmapsto \mathrm{diag}(\zeta^{q_1}, \dots, \zeta^{q_n}) \subseteq S^{2n-1}$

$\pi_1(G\backslash S^{2n-1}) = \langle \zeta \rangle$.

Circle:

$$\mathbb{R} \xrightarrow{p} S^1, \quad t \longmapsto e^{2\pi i t} \quad \Rightarrow \pi_1(S^1) = \mathrm{Deck}(p) \cong \mathbb{Z}.$$

Co/Homology:

Homology is a functor satisfying the Axioms:

$$(X,A) \longrightarrow \mathcal{A}_*$$

Pairs of top' spaces $A \subseteq X$ with morphisms cts maps $f:(X,A)\to(Y,B)$, $f(A)\subseteq B$.

Graded abelian group with homomorphisms.

H_* & H^* functoriality:

$f:(X,A) \longrightarrow (Y,B)$ morphism in $\mathcal{D}$

Then $H_*(f) = f_* : H_*(X,A) \longrightarrow H_*(Y,B)$ (covariant)

$H^*(f) = f^* : H^*(Y,B) \longrightarrow H^*(X,A)$ (contravariant)

$(fg)_* = f_* g_*$, $(fg)^* = g^* f^*$

$id_* = id_{H_*(-)}$, $id^* = id_{H^*(-)}$

> Ham-Sandwhich:
> Given n closed subsets of $\mathbb{R}^n$
> $\exists$ a hyperplane cutting each into two equal parts simultaneously.

T: It follows from the axioms that if $f: X \to Y$ is a homotopy equivalence then f_* & f^* are isomorphisms.

> Borsuk-Ulam:
> $f: S^n \to \mathbb{R}^n$ cts
> $\Rightarrow \exists x \quad f(x) = f(-x)$

> $H_*(\mathbb{RP}^n)$ from cellular:
>
> $\cdots 0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix}1 & -1\\ \vdots & \vdots\end{bmatrix}} \cdots \to \mathbb{Z}\oplus\mathbb{Z} \to 0$
>
> $\cdots 0 \to \mathbb{Z} \xrightarrow{[1, -(1)^{n+1}]} \cdots \to \mathbb{Z} \to 0$, $\times \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd}\end{cases}$
>
> $S^n \longrightarrow \mathbb{RP}^n$
>
> Two cells in each dim; one cell in each

Determining Which Functors: (Axioms)

There is more than one functor $\mathcal{D} \longrightarrow \mathcal{A}$, however we require co/homology to satisfy the Eilenberg-Steenrod Axioms, which uniquely determines a functor H_* / H^*:

- Natural transformation δ (Boundary map)

  We require a map $\delta : H_*(X,A) \longrightarrow H_{*-1}(A) = H(A,\emptyset)$

  such that the following commutes $\forall n \geq 1$ $\forall f:(X,A)\longrightarrow(Y,B)$

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y,B) \arrow[r, "\delta"] & H_{n-1}(B) \end{tikzcd}$$

For cohomology reverse the (δ) arrows

i.e. $\delta : H^n(A) \longrightarrow H^{n+1}(X,A)$

- Homotopy: $f, g : (X,A) \longrightarrow (Y,B)$ homotopic ($f \simeq g$)

  $\Rightarrow f_* = g_*$, $f^* = g^*$

- Excision: $\mathcal{U} \subseteq A$ open, $\overline{\mathcal{U}} \subseteq \mathring{A}$ (interior)

  $\Rightarrow i : (X\setminus\mathcal{U}, A\setminus\mathcal{U}) \hookrightarrow (X,A)$ (inclusion)

  induces isomorphism $i_* : H_*(X\setminus\mathcal{U}, A\setminus\mathcal{U}) \longrightarrow H_*(X,A)$

  $i^* : H^*(X,A) \longrightarrow H^*(X\setminus\mathcal{U}, A\setminus\mathcal{U})$

- Long Exact sequence: Recall this means the kernel of each map is the image of the previous map.

$$H_0(A) \xrightarrow{j_*} H_0(X) \xrightarrow{(id)_*} H_0(X,A)$$
$$\cdots$$
$$\xrightarrow{\delta} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \xrightarrow{\delta}$$
$$H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta}$$
$$\cdots$$

$(A,\emptyset) \overset{j}{\hookrightarrow} (X,\emptyset)$

- Dimension: $H_*(\text{point}) = H^*(\text{point}) = \begin{cases} \mathbb{Z}, & H_0 \\ 0, & H_n \ n>0. \end{cases}$

- Coproducts: Both H_* & H^* preserve arbitrary coproducts

  i.e. $\coprod_{\alpha\in A} X_\alpha \longmapsto H_n\left(\coprod_{\alpha\in A} X\right) = \coprod_{\alpha\in A} H_n(X_\alpha)$

  $\longmapsto H^n\left(\coprod_{\alpha\in A} X_\alpha\right) = \coprod_{\alpha\in A} H^n(X_\alpha)$

$f: X \longrightarrow Y$ a homotopy equivilence of spaces $\Rightarrow f^*$ & f_* are isomorphisms.

Reduced Homology: $H_*(X) \cong \underbrace{\tilde{H}_*(X)}_{\text{Reduced Homology}} \oplus H_*(pt)$

$$H_*(X,A) \cong H_*(\underbrace{X \cup_A \mathrm{Cone}(X)}_{\text{Mapping cone of } A \hookrightarrow X}, pt)$$

> Mayer-Vietoris: (X, x_0) and $X = \mathring{A} \cup \mathring{B}$, $x_0 \in (\mathring{A\cap B})$
>
> $$\cdots \xrightarrow{\delta} \tilde{H}_i(A\cap B) \xrightarrow{(k_*, l_*)} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \xrightarrow{i_* - j_*} \tilde{H}_i(X) \xrightarrow{\delta} \cdots$$
>
> $$\begin{tikzcd} & X & \\ A \arrow[ur, "i"] & & B \arrow[ul, "j"'] \\ & A\cap B \arrow[ul, "k"] \arrow[ur, "l"'] & \end{tikzcd}$$
>
> are inclusions
>
> Exact sequence

Axiomatic Reduced Homology:

$\tilde{H}_n : \underbrace{\mathcal{T}op_*}_{\text{Pointed top spaces}} \longrightarrow \underbrace{\mathcal{A}\mathcal{A}_*}_{\text{graded abelian groups}}$

1. Homotopy: $f \simeq g \Rightarrow f_* = g_*$ $(f(x_0) = g(x_0) = y_0)$
2. Additivity: $\bigvee_{\alpha\in A} X_\alpha \longmapsto \coprod_{\alpha\in A} \tilde{H}_n(X_\alpha)$
3. Mayer Vietoris (sequence exists)
4. Suspension: $H_*(\Sigma X; pt) \cong H_{*-1}(X, x_0)$
5. Dimension: $\tilde{H}_n(S^0) \cong \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n\geq 1 \end{cases}$

For a CW complex: $X^{(n)}/X^{(n-1)} \cong \bigvee_\alpha S^n$ over n cells in $X^{(n)}$

$X^{(n)}$: n skeleton

$$\begin{tikzcd} & H_n(X^{(n)}) \arrow[dr, "q_{n*}"] & & & \\ \tilde{H}_{n+1}(X^{(n+1)}/X^{(n)}) \arrow[ur, "\delta"] \arrow[r, "d_{n+1}"] & \tilde{H}_n(X^{(n)}/X^{(n-1)}) \arrow[r, "d_n"] \arrow[dr, "\delta"] & \tilde{H}_{n-1}(X^{(n-1)}/X^{(n-2)}) \\ & & H_{n-1}(X^{(n-1)}) \arrow[u, "q_{n-1*}"] \end{tikzcd}$$

$\leftarrow$ Quotient maps

$\tilde{H}_n(X^{(n)}/X^{(n-1)}) \cong C_n(X)$

$\cong \mathbb{Z}^{\#\text{ of n cells in } X}$

$H_n(X) = \ker(d_n)/\mathrm{Im}(d_{n+1})$

$d_n(\text{cell in } X) =$ degree of attaching map composed with collapses to wedge summands of $X^{(n-1)}/X^{(n-2)}$

$a \hookrightarrow X^{(n)} \xrightarrow{p} X^{(n-1)}/X^{(n-2)}$

Cellular Approx: Given X & Y CW complexes & any cts $f: X \to Y$

$\Rightarrow$ • $f \simeq$ cellular map

• Any two cellular maps are related by a cellular homotopy

$X^{(n)} \longrightarrow Y^{(n)} \quad \forall n$

Tells us our constructions are independent of the particular cell structure chosen.